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On semi-supermanifolds

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Abstract. Initially noninvertible objects are naturally arise in supermathematics, but it is common to deal with invertible ones only factoring former out in some extent. We propose to reconsider this ansatz and try to redefine such fundamental notions as supermanifolds, fiber bundles and homotopies using some weakening invertibility conditions. The prefix "semi-" reflects the fact that underlying morphisms form corresponding semigroups consisting of a known group part and a new ideal (disjoint) noninvertible part. We found that the absence of invertibility gives us some natural generalization of the cocycle conditions for transition functions of supermanifolds and fiber bundles. That can lead to construction of noninvertible analogs of Čech cocycles and spectral sequences. We also define semi-homotopies, which can be noninvertible and describe mappings into the semi-supermanifolds introduced.

1. Introduction

Noninvertible extension of the notion of a supermanifold seems intuitively natural in connection with the guesses made in the past concerning inner noninvertibility inherent in the supermanifold theory, e.g. "...a general SRS needs not have a body" [17], "...there may be no inverse projection (body map [79]) at all" [66], or "...a body may not even exist in the most extreme examples" [12]. In particular, while investigating noninvertible properties of superconformal symmetry [23] it was assumed [24] the possible existence of supersymmetric objects analogous to super Riemann surfaces, but without body, and shown preliminary how to construct them [27]. The superconformal semigroups arisen belong to a new class of semigroups having unusual abstract properties [28]. From other side, these investigations initiated studies of supermatrix semigroups [25] and supermatrix representations of semigroup bands [26].

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Noninvertibility in supermanifold theory [3, 87, 38] actually arises from odd nilpotent elements and zero divisors of underlying Grassmann-Banach algebras (see [42, 70, 83] for nontrivial examples).

In the infinite dimensional case there exist (topologically) quasinilpotent odd elements which are not really nilpotent [71], and, moreover, in some superalgebras one can construct pure soul elements which are not nilpotent even topologically [69] or introduce an invertible analog of an odd symbol [48], or use methods of nonstandard analysis [70]. Moreover, it were considered pure odd supermanifolds [76, 73], exotic supermanifolds with nilpotent even coordinates [43] and Frobenius supermanifolds with nonfixed metrics [57, 32], as well as supergravity with noninvertible vierbein [21].

We should also mention the possibility of defining a supermanifold without the notion of topological space [61]. Other supermanifold problems with odd directions (and therefore connected with noninvertibility in either event) are described in [3, 13, 14, 41], and a list of them is presented in [51].

2. **Standard patch definition of a supermanifold**

Let us remind the standard patch definition [74, 77, 87] of a supermanifold $\mathfrak{M}_0$ [79] (which differs from ordinary manifold definition [44, 46] by "super-" terminology only). We consider a collection of superdomains U_α such that $\mathfrak{M}_0 = \bigcup_\alpha U_\alpha$. Then in every superdomain U_α we take some superfunctions (*coordinate maps*)

$$\varphi_\alpha : U_\alpha \to D^{n|m} \subset \mathbb{R}^{n|m}, \tag{1}$$

where $\mathbb{R}^{n|m}$ is a superspace in which our super "ball" lives, and $D^{n|m}$ is an open domain in $\mathbb{R}^{n|m}$. Next we call the pair $\{U_\alpha, \varphi_\alpha\}$ a *local chart* and claim that the union of charts $\bigcup_\alpha \{U_\alpha, \varphi_\alpha\}$ is an atlas of a *supermanifold* [20, 79, 87].

Next we introduce gluing *transition functions* as follows. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \varnothing$ and

$$\begin{aligned} \varphi_\alpha &: U_\alpha \to V_\alpha \subset \mathbb{R}^{n|m}, \\ \varphi_\beta &: U_\beta \to V_\beta \subset \mathbb{R}^{n|m}. \end{aligned} \tag{2}$$

Then the above morphisms are restricted to $\varphi_\alpha : U_{\alpha\beta} \to V_{\alpha\beta} = V_\alpha \cap \varphi_\alpha (U_{\alpha\beta})$ and $\varphi_\beta : U_{\alpha\beta} \to V_{\beta\alpha} = V_\beta \cap \varphi_\beta (U_{\alpha\beta})$. The maps $\Phi_{\alpha\beta} : V_{\beta\alpha} \to V_{\alpha\beta}$ which are called to make the following diagram

$$\begin{array}{ccc} U_{\alpha\beta} & \xrightarrow{\varphi_\beta} & V_{\beta\alpha} \\ & \searrow^{\varphi_\alpha} & \downarrow \Phi_{\alpha\beta} \\ & & V_{\alpha\beta} \end{array} \tag{3}$$

to be commuted are named transition functions of a supermanifold in a given atlas. Here we stress, first, that $U_{\alpha\beta} \subset \mathfrak{M}_0$ and $V_{\alpha\beta}, V_{\beta\alpha} \subset \mathbb{R}^{n|m}$. Second, from (3) one usually concludes that

$$\Phi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} \tag{4}$$

The transition superfunctions $\Phi_{\alpha\beta}$ give us possibility to restore the whole supermanifold from individual charts and coordinate maps. Indeed they contain all information about the supermanifold. They may belong to different functional classes, which gives possibility to specify more narrow classes of manifolds and supermanifolds, for instance (super)smooth, analytic, Lipschitz and others [44, 65]. Mostly the prefix "super-" only distinguishes the patch definitions of a manifold and supermanifold (which gives us possibility to write it in brackets) and the properties of $\Phi_{\alpha\beta}$ [13, 79, 20]. Here we do not discuss them in detail and try to put minimum restrictions on $\Phi_{\alpha\beta}$, concentrating our attention on their abstract properties and generalizations following from them.

Additionally, from (4) it follows that transition functions satisfy the cocycle conditions

$$\Phi_{\alpha\beta}^{-1} = \Phi_{\beta\alpha} \tag{5}$$

on $U_\alpha \cap U_\beta$ and

$$\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} = 1_{\alpha\alpha} \tag{6}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$, where $1_{\alpha\alpha} \stackrel{\text{def}}{=} \text{id}\,(U_\alpha)$.

Usually [79, 74, 87] it is implied that all the maps φ_α are homeomorphisms, and they can be described by one-to-one invertible continuous (super)smooth functions (i.e. one wants "to jump" in both directions between any two intersecting domains $U_\alpha \cap U_\beta \neq \varnothing$). First, it was reasonable not to distinguish between U_α and $D^{n|m}$, i.e. locally supermanifolds are as the whole superspace $\mathbb{R}^{n|m}$ [20, 41]. However the matter is not only in more rich fiber bundle [10, 11, 30] and sheaf [45, 50, 49] structures due to consideration of all constructions over underlying Grassmann algebra (or more general ones [69, 83, 87, 39]).

The problem lies in another abstract level of the constructions, if the invertibility conditions are weakened in some extent (see, e.g. [23, 27]).

3. Noninvertible extension of a supermanifold

Earlier there was the following common prescription: one had ready objects (e.g. real manifolds which can be investigated almost visually [15, 22, 81]), and then using various methods and guesses one found restrictions on transition functions. Notwithstanding, noninvertible functions were simply excluded (saying magic words "factorizing by nilpotents we again derive the well-known result") from consideration, because of desire to be in the nearest analogy with intuitively clear and understandable nonsupersymmetric case [44, 46, 65].

Here we go in opposite direction: we know that in supermathematics noninvertible variables and functions *do exist*. Which objects could be constructed by means of them? What gives "factorizing by non-nilpotents", i.e. consideration of non-group features of theory? How changes the general abstract sense of the most important notions, e.g. manifolds and fiber bundles? We now try to leave aside inner structure of noninvertible objects analogous to supermanifolds and concentrate our attention on there general abstract definitions.

We note that among ordinary functions there exist noninvertible ones as well [54, 55], but the kind of noninvertibility considered here is very special: it appears only due to the existence of nilpotents in underlying superalgebra [42, 71, 83]. So that, the noninvertible extension of a manifold proposed below is very special and can exist due to the presence of supersymmetry only.

Here we do not consider concrete equations and methods of their solving, we only use the fact of their existence to reformulate some definitions and extend well-known notions.

3.1. **Definition of a semi-supermanifold.** Now we formulate a patch definition of an object analogous to supermanifold, i.e. try to weaken demand of invertibility of coordinate maps (2). Let us consider a generalized superspace $\mathfrak{M}$ covered by open sets U_α as $\mathfrak{M} = \bigcup\limits_\alpha U_\alpha$. We assume here that the maps $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{n|m}$ are not all homeomorphisms, i.e. among them there are noninvertible maps[1].

Definition 1. A *chart* is a pair $\{U_\alpha^{inv}, \varphi_\alpha^{inv}\}$,where φ_α^{inv} is an invertible morphism. A *semi-chart* is a pair $\{U_\alpha^{noninv}, \varphi_\alpha^{noninv}\}$, where φ_α^{noninv} is a noninvertible morphism.

[1] Indeed in this sense the superspace $\mathfrak{M}$ is noninvertibly generalized, and instead of $\mathbb{R}^{n|m}$ one can consider some its noninvertible generalization.

Definition 2. A *semi-atlas* $\{U_\alpha, \varphi_\alpha\}$ is a union of charts and semi-charts

$$\left\{U_\alpha^{inv}, \varphi_\alpha^{inv}\right\} \bigcup \left\{U_\alpha^{noninv}, \varphi_\alpha^{noninv}\right\}. \tag{7}$$

Definition 3. A *semi-supermanifold* is a noninvertibly generalized superspace $\mathfrak{M}$ represented as a semi-atlas $\mathfrak{M} = \bigcup\limits_\alpha \{U_\alpha, \varphi_\alpha\}$.

How to define an analog of transition function? We should consider the same diagram (3), but we cannot use (4) through noninvertibility of some of φ_α's.

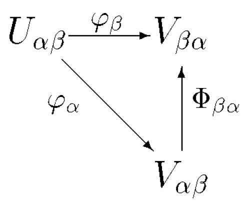
Definition 4. Gluing *semi-transition functions* of a semi-supermanifold are defined by the equations

$$\Phi_{\alpha\beta} \circ \varphi_\beta = \varphi_\alpha, \tag{8}$$

$$\Phi_{\beta\alpha} \circ \varphi_\alpha = \varphi_\beta. \tag{9}$$

We stress that to determine $\Phi_{\alpha\beta}$ the equation (8) cannot be solved by (4) in noninvertible case[2]. Instead we should find artificial methods of its solving, e.g. as in previous subsection, expanding in superalgebra generator series, or using abstract semigroup methods [36, 33, 52] and consider solutions of noninvertible equations as equivalence classes.

The semi-transition functions entering in (8) and (9) are not more mutually inverse, and the functions $\Phi_{\beta\alpha}$ should now be determined not from (5) in which the left hand side is not well defined, but from the commutative diagram



(10)

and the equation (9) following from it. They are also can be noninvertible and therefore the cocycle conditions should be modified not to use invertible functions only.

Remark. Even in the standard case the cocycle conditions (6) for supermanifolds are not automatically satisfied when (4) holds, and therefore they should be imposed by hand [64].

[2]A semigroup analog of that is the difference between inverse and regular elements [35, 33].

Thus, instead of (5) and (6) we have

Conjecture 5. *The semi-transition functions $\Phi_{\alpha\beta}$ of a semi-supermanifold satisfy the following relations*

$$\Phi_{\alpha\beta}\circ\Phi_{\beta\alpha}\circ\Phi_{\alpha\beta}=\Phi_{\alpha\beta} \tag{11}$$

on $U_\alpha\cap U_\beta$ overlaps and

$$\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}\circ\Phi_{\gamma\alpha}\circ\Phi_{\alpha\beta}=\Phi_{\alpha\beta}, \tag{12}$$

$$\Phi_{\beta\gamma}\circ\Phi_{\gamma\alpha}\circ\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}=\Phi_{\beta\gamma}, \tag{13}$$

$$\Phi_{\gamma\alpha}\circ\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}\circ\Phi_{\gamma\alpha}=\Phi_{\gamma\alpha} \tag{14}$$

on triple overlaps $U_\alpha\cap U_\beta\cap U_\gamma$ and

$$\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}\circ\Phi_{\gamma\rho}\circ\Phi_{\rho\alpha}\circ\Phi_{\alpha\beta}=\Phi_{\alpha\beta}, \tag{15}$$

$$\Phi_{\beta\gamma}\circ\Phi_{\gamma\rho}\circ\Phi_{\rho\alpha}\circ\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}=\Phi_{\beta\gamma}, \tag{16}$$

$$\Phi_{\gamma\rho}\circ\Phi_{\rho\alpha}\circ\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}\circ\Phi_{\gamma\rho}=\Phi_{\gamma\rho}, \tag{17}$$

$$\Phi_{\rho\alpha}\circ\Phi_{\alpha\beta}\circ\Phi_{\beta\gamma}\circ\Phi_{\gamma\rho}\circ\Phi_{\rho\alpha}=\Phi_{\rho\alpha} \tag{18}$$

on $U_\alpha\cap U_\beta\cap U_\gamma\cap U_\rho$.

Here the first relation (11) is called to generalize the first cocycle condition (5), while other relations correspond (6).

Definition 6. We call (11)–(18) *tower relations.*

Definition 7. A semi-supermanifold is called *reflexive* if, in addition to (11)–(18), the semi-transition functions satisfy to the reflexivity conditions

$$\Phi_{\beta\alpha}\circ\Phi_{\alpha\beta}\circ\Phi_{\beta\alpha}=\Phi_{\beta\alpha} \tag{19}$$

on $U_\alpha\cap U_\beta$ overlaps and

$$\Phi_{\alpha\gamma}\circ\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}\circ\Phi_{\alpha\gamma}=\Phi_{\alpha\gamma}, \tag{20}$$

$$\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}\circ\Phi_{\alpha\gamma}\circ\Phi_{\gamma\beta}=\Phi_{\gamma\beta}, \tag{21}$$

$$\Phi_{\beta\alpha}\circ\Phi_{\alpha\gamma}\circ\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}=\Phi_{\beta\alpha} \tag{22}$$

on triple overlaps $U_\alpha\cap U_\beta\cap U_\gamma$ and

$$\Phi_{\alpha\rho}\circ\Phi_{\rho\gamma}\circ\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}\circ\Phi_{\alpha\rho}=\Phi_{\alpha\rho}, \tag{23}$$

$$\Phi_{\rho\gamma}\circ\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}\circ\Phi_{\alpha\rho}\circ\Phi_{\rho\gamma}=\Phi_{\rho\gamma}, \tag{24}$$

$$\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}\circ\Phi_{\alpha\rho}\circ\Phi_{\rho\gamma}\circ\Phi_{\gamma\beta}=\Phi_{\gamma\beta}, \tag{25}$$

$$\Phi_{\beta\alpha}\circ\Phi_{\alpha\rho}\circ\Phi_{\rho\gamma}\circ\Phi_{\gamma\beta}\circ\Phi_{\beta\alpha}=\Phi_{\beta\alpha} \tag{26}$$

on $U_\alpha\cap U_\beta\cap U_\gamma\cap U_\rho$.

Remark. One could think that the reflexivity conditions differ from (11)–(18) by index permutations only. This is true. But the functions $\Phi_{\alpha\beta}$ entering in (11)–(18) and in (19)–(26) *are the same*, and therefore the latter are independent equations imposed on $\Phi_{\alpha\beta}$.

Remark. In any actions with noninvertible functions $\Phi_{\alpha\beta}$ we are not allowed to cancel, because the semigroup of $\Phi_{\alpha\beta}$'s is a semigroup without cancellation, and we are forced to exploit the corresponding methods [18, 34, 56, 86].

Corollary 8. *The relations (11)–(26) satisfy identically in the standard invertible case* [20, 41, 87, 79]*, i.e. when the conditions (4), (5) and (6) hold valid.*

Remark. The equations (8)–(9) defining the semi-transition functions $\Phi_{\alpha\beta}$ can have no unique solutions, and so in that case $\Phi_{\alpha\beta}$ should be considered as corresponding sets of functions.

Conjecture 9. *The functions $\Phi_{\alpha\beta}$ satisfying the relations (11)–(26) can be viewed as some noninvertible generalization of the transition functions as cocycles in the Čech cohomology of coverings* [53, 84]*.*

3.2. **Orientation of semi-supermanifolds.** It is well known that orientation of ordinary manifolds is determined by the Jacobian sign of transition functions $\Phi_{\alpha\beta}$ written in terms of local coordinates on $U_\alpha \cap U_\beta$ overlaps [44, 46]. Since this sign belong to $\mathbb{Z}_2$, there exist two orientations on U_α. Two overlapping charts are *consistently oriented* (or *orientation preserving*) if $\Phi_{\alpha\beta}$ has positive Jacobian, and a manifold is *orientable* if it can be covered by such charts, thus there are two kinds of manifolds: orientable and nonorientable [44, 46]. In supersymmetric case the role of Jacobian plays Berezinian [7] which has a "sign" belonging to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [88, 85], and so there are four orientations on U_α and five corresponding kinds of supermanifold orientability [61, 82].

Definition 10. In case a nonvanishing Berezinian of $\Phi_{\alpha\beta}$ is nilpotent (and so has no definite sign in the previous sense) there exists additional *nilpotent orientation* on U_α of a semi-supermanifold.

A degree of nilpotency of Berezinian allows us to classify semi-supermanifolds having nilpotent orientability.

3.3. **Obstructedness and semi-supermanifolds.** The semi-supermanifolds defined above belong to a class of so called obstructed semi-supermanifolds in the following sense. Let us rewrite (4), (5) and (6) as the infinite series

$$n = 1 : \quad \Phi_{\alpha\alpha} = 1_{\alpha\alpha}, \tag{27}$$

$$n = 2: \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = 1_{\alpha\alpha}, \tag{28}$$

$$n = 3: \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} = 1_{\alpha\alpha}, \tag{29}$$

$$n = 4: \quad \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\delta} \circ \Phi_{\delta\alpha} = 1_{\alpha\alpha} \tag{30}$$

$$\ldots \quad \ldots$$

Definition 11. A semi-supermanifold is called *obstructed* if some of the cocycle conditions (27)–(30) are broken.

Remark. The introduced notion of obstructed manifold should not be mixed with the notion of obstruction for ordinary manifolds [5] and supermanifolds [7] or obstruction to extensions [53] and in the theory of characteristic classes [60, 40].

In some cases it can happen that starting from some $n = n_m$ all higher cocycle conditions hold valid.

Definition 12. *Obstructedness degree* of a semi-supermanifold is a maximal n_m for which the cocycle conditions (27)–(30) are broken. If all of them hold valid, then $n_m \stackrel{\text{def}}{=} 0$.

Corollary 13. *Ordinary manifolds (with invertible transition functions) have vanishing obstructedness, and the obstructedness degree for them is equal to zero, i.e.* $n_m = 0$*.*

Conjecture 14. *The obstructed semi-supermanifolds may have non-vanishing ordinary obstruction which can be calculated extending the standard methods* [7] *to the non-invertible case.*

Therefore, using the obstructedness degree n_m, we have possibility to classify semi-supermanifolds properly.

Remark. In general, semi-supermanifolds can be defined by algebra of semi-transition functions only.

In search of some analogies we can compare semi-supermanifolds with supernumbers as follows

Supernumbers	**Semi-supermanifolds**
Ordinary nonzero numbers (invertible)	Ordinary manifolds (transition functions are invertible)
Supernumbers having a nonvanishing body part (invertible)	Supermanifolds (transition functions are invertible)
Pure soul supernumbers without a body part (noninvertible)	Obstructed semi-supermanifolds (transition functions are noninvertible)

Further consideration concerns the well-known fact [20, 58] that the pure soul supernumbers do exist only in the presence of odd nilpotent directions [68, 74, 75].

Remark. Obstructed semi-supermanifolds have nonzero odd dimension.

Moreover, obviously the pure soul supernumbers do not contain unity.

Remark. Obstructed semi-supermanifolds cannot have identity semi-transition functions.

As possible example of semi-transition functions of obstructed semi-supermanifolds one can consider the twisting parity of tangent space transformations introduced in [23, 27]. Objects obtained in this way can be viewed as noninvertible analogs of super Riemann surfaces [17], which will be investigated in more detail elsewhere.

3.4. **Tower identity semigroup.** Let us consider a series of the self-maps $\mathrm{e}_{\alpha\alpha}^{(n)} : U_\alpha \to U_\alpha$ of a semi-supermanifold defined as

$$\mathrm{e}_{\alpha\alpha}^{(1)} = \Phi_{\alpha\alpha}, \tag{31}$$

$$\mathrm{e}_{\alpha\alpha}^{(2)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha}, \tag{32}$$

$$\mathrm{e}_{\alpha\alpha}^{(3)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha}, \tag{33}$$

$$\mathrm{e}_{\alpha\alpha}^{(4)} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\delta} \circ \Phi_{\delta\alpha} \tag{34}$$

$$\cdots \quad \cdots$$

Definition 15. We will call $e^{(n)}_{\alpha\alpha}$'s *tower identities.*

From (27)–(30) it follows

Assertion 16. *For ordinary supermanifolds all tower identities coincide with the usual identity map*

$$e^{(n)}_{\alpha\alpha} = 1_{\alpha\alpha}. \tag{35}$$

Remark. In the trivial case, when all $\Phi_{\alpha\beta}$ are identity maps, it is obvious that the relations (31)-(34) satisfy identically.

The obstructedness degree can be treated as a maximal $n = n_m$ for which tower identities differ from the identity, i.e. (35) is broken. So the tower identities give the numerical measure of distinction of a semi-supermanifold from an ordinary supermanifold. Being an important inner characteristic the tower identities (31)–(34) play a deep fundamental role in description of semi-supermanifolds. Therefore, we will study some of their properties in detail.

Proposition 17. *The tower identities are units for the semi-transition functions*

$$e^{(n)}_{\alpha\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}, \tag{36}$$

$$\Phi_{\alpha\beta} \circ e^{(n)}_{\beta\beta} = \Phi_{\alpha\beta}. \tag{37}$$

Proof. It follows directly from the tower relations (11)–(18) and the definition (11)–(18). □

Proposition 18. *The tower identities are idempotents*

$$e^{(n)}_{\alpha\alpha} \circ e^{(n)}_{\alpha\alpha} = e^{(n)}_{\alpha\alpha}. \tag{38}$$

Proof. We prove the statement for $n = 2$ and for other n it can be proved by induction. We write (38) as

$$e^{(2)}_{\alpha\alpha} \circ e^{(2)}_{\alpha\alpha} = e^{(2)}_{\alpha\alpha} \circ \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \left(e^{(2)}_{\alpha\alpha} \circ \Phi_{\alpha\beta}\right) \circ \Phi_{\beta\alpha}.$$

Then using (36) we obtain

$$\left(e^{(2)}_{\alpha\alpha} \circ \Phi_{\alpha\beta}\right) \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = e^{(2)}_{\alpha\alpha}.$$

□

Remark. The functional nonsupersymmetric equations of the above kind were studied in [6].

Definition 19. *Conjugate tower identities* correspond to the same partition of the semi-supermanifold and consists of the semi-transition functions taken in opposite order

$$\tilde{\mathrm{e}}^{(1)}_{\alpha\alpha} = \mathrm{e}^{(1)}_{\alpha\alpha}, \tag{39}$$

$$\tilde{\mathrm{e}}^{(2)}_{\alpha\alpha} = \mathrm{e}^{(2)}_{\alpha\alpha}, \tag{40}$$

$$\tilde{\mathrm{e}}^{(3)}_{\alpha\alpha} = \Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha}, \tag{41}$$

$$\tilde{\mathrm{e}}^{(4)}_{\alpha\alpha} = \Phi_{\alpha\delta} \circ \Phi_{\delta\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} \tag{42}$$

$$\ldots \quad \ldots$$

The conjugate tower identities play the role of tower identities, but for reflexivity conditions (19)–(26). By analogy with (36)–(37) we have

Proposition 20. *The conjugate tower identities are reflexive units for the semi-transition functions*

$$\tilde{\mathrm{e}}^{(n)}_{\beta\beta} \circ \Phi_{\beta\alpha} = \Phi_{\beta\alpha}, \tag{43}$$

$$\Phi_{\beta\alpha} \circ \tilde{\mathrm{e}}^{(n)}_{\alpha\alpha} = \Phi_{\beta\alpha}. \tag{44}$$

Proposition 21. *For the same partition the conjugate tower identities annihilate the tower identities in the following sense*

$$\mathrm{e}^{(n)}_{\alpha\alpha} \circ \tilde{\mathrm{e}}^{(n)}_{\alpha\alpha} = \mathrm{e}^{(2)}_{\alpha\alpha}. \tag{45}$$

Proof. Let us consider the case $n = 3$. Using the definitions we derive

$$\mathrm{e}^{(3)}_{\alpha\alpha} \circ \tilde{\mathrm{e}}^{(3)}_{\alpha\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\alpha} \circ \Phi_{\alpha\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha}$$

$$= \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ (\Phi_{\gamma\alpha} \circ \Phi_{\alpha\gamma}) \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \mathrm{e}^{(2)}_{\gamma\gamma} \circ \Phi_{\gamma\beta} \circ \Phi_{\beta\alpha}$$

$$= \Phi_{\alpha\beta} \circ (\Phi_{\beta\gamma} \circ \Phi_{\gamma\beta}) \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \mathrm{e}^{(2)}_{\beta\beta} \circ \Phi_{\beta\alpha} = \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \mathrm{e}^{(2)}_{\alpha\alpha}.$$

For other n the statement can be proved by induction. □

Definition 22. A semi-supermanifold is *nice* (see, e.g. [2]), if the tower identities do not depend on a given partition.

The multiplication of the tower identities of a nice semi-supermanifold can be defined as follows

$$\mathrm{e}^{(n)}_{\alpha\alpha} \circ \mathrm{e}^{(m)}_{\alpha\alpha} = \mathrm{e}^{(n+m)}_{\alpha\alpha}. \tag{46}$$

Assertion 23. *The multiplication (46) is associative.*

Therefore, we are able to give

Definition 24. The tower identities of a nice semi-supermanifold form a *tower semigroup* under the multiplication (46).

So we obtained a quantitative description of inner noninvertibility properties of semi-supermanifolds.

Conjecture 25. *The introduced tower semigroup plays the same principal role for semi-supermanifolds as the fundamental group for ordinary manifolds* [29, 53, 84].

3.5. **Semi-commutative diagrams and n-regularity.** The above constructions have the general importance for any set of noninvertible mappings.

Remark. The extension of $n = 2$ cocycle given by (11) can be viewed as some analogy with regular [31, 1, 16] or pseudoinverse [62] elements in semigroups or generalized inverses in matrix theory [59, 67, 78], category theory [19] and theory of generalized inverses of morphisms [63].

The relations (12)–(18) and with other n can be considered as noninvertible analogue of regularity for higher cocycles. Therefore, by analogy with (11)–(18) it is natural to formulate the general

Definition 26. An noninvertible mapping $\Phi_{\alpha\beta}$ is *n-regular*, if it satisfies to the following conditions

$$\overbrace{\Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \ldots \circ \Phi_{\rho\alpha} \circ \Phi_{\alpha\beta}}^{n+1} = \Phi_{\alpha\beta} + \textit{permutations} \tag{47}$$

on overlaps $\overbrace{U_\alpha \cap U_\beta \cap \ldots \cap U_\rho}^{n}$.

The formula (11) describes 3-regular mappings, the relations (12)–(14) correspond to 4-regular ones, and (15)–(18) give 5-regular mappings.

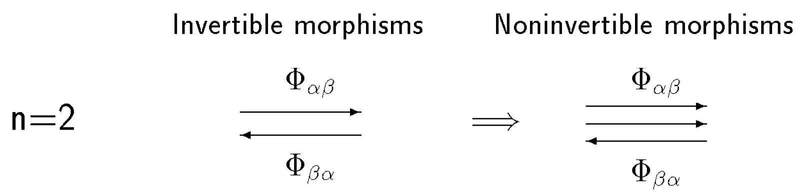
Remark. The 3-regularity coincides with the ordinary regularity.

Another definition of n-regularity can be given by the formulas (36)–(37).

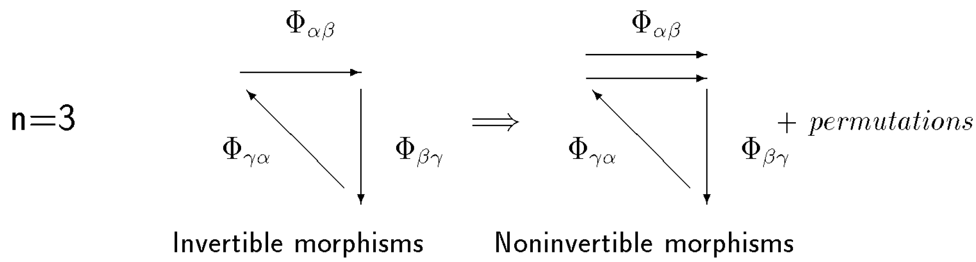
The higher regularity conditions change dramatically the general diagram technique of morphisms, when we turn to noninvertible ones. Indeed, the commutativity of invertible morphism diagrams is based on the relations (27)–(30), i.e. on the fact that the tower identities are ordinary identities (35). When morphisms are noninvertible (a semi-supermanifold has a nonvanishing obstructedness), we cannot "return to the same point", because in general $\mathrm{e}_{\alpha\alpha}^{(n)} \neq 1_{\alpha\alpha}$, and we have to consider "nonclosed" diagrams due to the fact that the relation $e^{(n)}_{\alpha\alpha} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\beta}$ is noncancellative now.

Summarizing the above statements we propose the following intuitively consistent changing of the standard diagram technique as applied to noninvertible morphisms. In every case we get a new arrow which corresponds to the additional multiplier in (36).

Thus, for $n = 2$ we obtain



which describes the transition from (28) to (11) and presents the ordinary regularity condition for morphisms [19, 63]. The most intriguing semicommutative diagram is the triangle one



which generalizes the cocycle condition (6).

By analogy we can write higher n-regularity semicommutative diagrams, which can be considered in framework of generalized categories [9, 47].

4. Noninvertibility and semi-superbundles

A similar approach can be applied to the noninvertible extension of super fiber bundles [30, 4], while defining them globally in terms of open coverings and transition functions [10].

Following the standard definitions [60, 65, 37] and weakening invertibility we now construct new objects analogous to fiber bundles.

4.1. **Definition of semi-superbundles.** Let E and $\mathfrak{M}$ be a *total* (*bundle*) superspace and *base* semi-supermanifold respectively, and $\pi : E \to \mathfrak{M}$ be a *semi-projection map* which is not necessarily invertible (but can be smooth). Denote by F_b the set of points of E that are mapped to $b \in \mathfrak{M}$ (a *pre-image* of b), i.e. the *semi-fiber over* b is $F_b \stackrel{\text{def}}{=} \{x \in E \,|\, \pi(x) = b\}$. Then, $F = \bigcup F_b$ is a *semi-fiber*.

Definition 27. A *semi-superbundle* is $\mathcal{L} \stackrel{\text{def}}{=} (E, \mathfrak{M}, F, \pi)$.

A *section* $\mathbf{s} : \mathfrak{M} \to F$ of the fiber bundle $(E, \mathfrak{M}, F, \pi)$ is usually defined by $\pi(\mathbf{s}(b)) = b$ which in the form $\pi \circ \mathbf{s} = 1_{\mathfrak{M}}$ is very similar to (5) and (28) and holds valid for invertible maps π and $\mathbf{s}$ only. Therefore, a very few ordinary nontrivial fiber bundles admit corresponding sections [60].

Thus, using analogy with (11), we come to the following

Definition 28. A *semi-section* of the semi-superbundle $\mathcal{L} = (E, \mathfrak{M}, F, \pi)$ is defined by

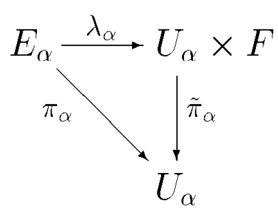
$$\pi \circ \mathbf{s} \circ \pi = \pi. \tag{48}$$

A *reflexive* semi-section satisfies to the additional condition

$$\mathbf{s}_{refl} \circ \pi \circ \mathbf{s}_{refl} = \mathbf{s}_{refl}. \tag{49}$$

Let $\tilde{\pi} : \mathfrak{M} \times F \to \mathfrak{M}$ is the canonical semi-projection on the first factor $\tilde{\pi}(b, f) = b,\ f \in F$, then $\tilde{\pi}$ gives rise to a *product fiber bundle*. If $\lambda : E \to \mathfrak{M} \times F$ is a morphism (called a *trivialization*), then $\tilde{\pi} \circ \lambda = \pi$, and the semi-superbundle $\mathcal{L} = (E, \mathfrak{M}, F, \pi)$ is *trivial*. If there exists a continuous map $\eta : \mathfrak{M} \to F$, then the semi-superbundle $(\mathfrak{M} \times F, \mathfrak{M}, F, \tilde{\pi})$ admits the section $\mathbf{s} : \mathfrak{M} \to \mathfrak{M} \times F$ given by $\mathbf{s}(b) = (b, \eta(b))$.

Let $E_\alpha \stackrel{\text{def}}{=} \{x \in E \,|\, \pi_\alpha(x) = b,\, b \in U_\alpha \subset \mathfrak{M}\}$ (here we do not use the standard notion $\pi^{-1}(U_\alpha)$ for E_α intentionally, because now π_α is allowed to be noninvertible), where $\pi_\alpha : E_\alpha \to U_\alpha$ is a restriction, i.e. $\pi_\alpha \stackrel{\text{def}}{=} \pi \,|_{U_\alpha}$. Then the semi-superbundle $\mathcal{L} = (E, \mathfrak{M}, F, \pi)$, is *locally trivial*, if $\forall b \in \mathfrak{M}\, \exists U_\alpha \ni b$ such that there exists the trivializing morphisms $\lambda_\alpha : E_\alpha \to U_\alpha \times F$ satisfying $\tilde{\pi}_\alpha \circ \lambda_\alpha = \pi_\alpha$. That is, the diagram



(50)

commutes.

Definition 29. A *semi-section* $\mathbf{s}_\alpha$ of a locally trivial semi-superbundle $\mathcal{L}$ is given by the maps $\mathbf{s}_\alpha : U_\alpha \to E$ which satisfy the compatibility conditions

$$\lambda_\alpha \circ \mathbf{s}_\alpha \mid_b = \lambda_\beta \circ \mathbf{s}_\beta \mid_b, \;\; b \in U_\alpha \cap U_\beta. \tag{51}$$

Now let $\{U_\alpha, \lambda_\alpha\}$ be a trivializing covering of π such that $\bigcup\limits_\alpha U_\alpha = \mathfrak{M}$ and $U_\alpha \cap U_\beta \neq \varnothing \Rightarrow E_\alpha \cap E_\beta \neq \varnothing$. Then we demand the trivializing morphisms λ_α to be agree, which means that the diagrams

$$\begin{array}{ccc} E_\alpha \cap E_\beta & \xrightarrow{\lambda_\beta} & U_\alpha \cap U_\beta \times F \\ & \searrow^{\lambda_\alpha} & \downarrow{\scriptstyle \Lambda_{\alpha\beta}} \\ & & U_\alpha \cap U_\beta \times F \end{array} \tag{52}$$

and

$$\begin{array}{ccc} E_\alpha \cap E_\beta & \xrightarrow{\lambda_\beta} & U_\alpha \cap U_\beta \times F \\ & \searrow^{\lambda_\alpha} & \uparrow{\scriptstyle \Lambda_{\beta\alpha}} \\ & & U_\alpha \cap U_\beta \times F \end{array} \tag{53}$$

should commute, where $\Lambda_{\alpha\beta}$ and $\Lambda_{\beta\alpha}$ are maps acting along a semi-fiber F.

Definition 30. Gluing *semi-transition functions* $\Lambda_{\alpha\beta}$ of a locally trivial semi-superbundle $\mathcal{L} = (E, \mathfrak{M}, F, \pi)$ are defined by the equations

$$\Lambda_{\alpha\beta} \circ \lambda_\beta = \lambda_\alpha, \tag{54}$$

$$\Lambda_{\beta\alpha} \circ \lambda_\alpha = \lambda_\beta. \tag{55}$$

Conjecture 31. *The semi-transition functions of a semi-superbundle $\mathcal{L}$ satisfy the following relations*

$$\Lambda_{\alpha\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta} \tag{56}$$

on $U_\alpha \cap U_\beta$ overlaps and

$$\Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\alpha} \circ \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}, \tag{57}$$

$$\Lambda_{\beta\gamma} \circ \Lambda_{\gamma\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{58}$$

$$\Lambda_{\gamma\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\alpha} = \Lambda_{\gamma\alpha} \tag{59}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ *and*

$$\Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \Lambda_{\rho\alpha} \circ \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}, \tag{60}$$

$$\Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \Lambda_{\rho\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{61}$$

$$\Lambda_{\gamma\rho} \circ \Lambda_{\rho\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} = \Lambda_{\gamma\rho}, \tag{62}$$

$$\Lambda_{\rho\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \Lambda_{\rho\alpha} = \Lambda_{\rho\alpha} \tag{63}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho$ *.*

Definition 32. A semi-superbundle $\mathcal{L}$ is called *reflexive* if, in addition to (56)-(63), the semi-transition functions satisfy to the reflexivity conditions

$$\Lambda_{\beta\alpha} \circ \Lambda_{\alpha\beta} \circ \Lambda_{\beta\alpha} = \Lambda_{\beta\alpha} \tag{64}$$

on $U_\alpha \cap U_\beta$ overlaps and

$$\Lambda_{\alpha\gamma} \circ \Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\gamma} = \Lambda_{\alpha\gamma}, \tag{65}$$

$$\Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\gamma} \circ \Lambda_{\gamma\beta} = \Lambda_{\gamma\beta}, \tag{66}$$

$$\Lambda_{\beta\alpha} \circ \Lambda_{\alpha\gamma} \circ \Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} = \Lambda_{\beta\alpha} \tag{67}$$

on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ and

$$\Lambda_{\alpha\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\rho} = \Lambda_{\alpha\rho}, \tag{68}$$

$$\Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\rho} \circ \Lambda_{\rho\gamma} = \Lambda_{\rho\gamma}, \tag{69}$$

$$\Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} \circ \Lambda_{\alpha\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} = \Lambda_{\gamma\beta}, \tag{70}$$

$$\Lambda_{\beta\alpha} \circ \Lambda_{\alpha\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \Lambda_{\beta\alpha} = \Lambda_{\beta\alpha} \tag{71}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho$.

For a fixed $b \in U_\alpha \cap U_\beta$ gluing transition function $\Lambda_{\alpha\beta}$ describe morphisms of a semi-fiber F to itself by the condition

$$\Lambda_{\alpha\beta} : (b, f) \to (b, L_{\alpha\beta} f)\,, \tag{72}$$

where $L_{\alpha\beta} : U_\alpha \cap U_\beta \to F$ and $f \in F$. The functions $L_{\alpha\beta}$ satisfy to the generalized cocycle conditions similar to (56)–(71).

Remark. It is well-known that sections and transition functions of an ordinary fiber bundle [37, 46] and superbundle [85] can be noninvertible even in the standard case. But that kind of noninvertibility has principally different nature comparing, as, e.g. (super)manifolds and semi-supermanifolds introduced above.

Remark. It can be compared with the possible noninvertibility of ordinary functions [55, 80] and the noninvertibility of superfunctions which takes place due to the presence of nilpotents and zero divisors.

The standard transition functions are implied to be homeomorphisms and sections should be in 1-1 correspondence with maps from the base to the fiber [72, 75]. Our definitions (11)-(26) and (48)-(71) extend them, allowing to include in consideration properly noninvertible superfunctions as well.

4.2. **Morphisms of semi-superbundles.** Let $\mathcal{L} = (E, \mathfrak{M}, F, \pi)$ and $\mathcal{L}' = (E', \mathfrak{M}', F', \pi')$ be two semi-superbundles.

Definition 33. A semi-superbundle morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ consists of two morphisms $f = (f_E, f_{\mathfrak{M}})$, where $f_E : E \to E'$ and $f_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}'$,satisfying $f_{\mathfrak{M}} \circ \pi = \pi' \circ f_E$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ \mathfrak{M} & \xrightarrow{f_{\mathfrak{M}}} & \mathfrak{M}' \end{array} \tag{73}$$

is commutative.

Let $E_b = \{x \in E \,|\, \pi(x) = b,\, b \in U \subset \mathfrak{M}\}$, then $f_E(E_b) \subset E'_{f_{\mathfrak{M}}(b)}$ for each b, and so the semi-fiber over $b \in \mathfrak{M}$ is carried into the semi-fiber over $f(b) \in \mathfrak{M}'$ by f_E being a fiber morphism. If a semi-superbundle has a section, f_E acts as follows $\mathbf{s}(b) \to \mathbf{s}'(f_{\mathfrak{M}}(b))$.

In most applications of fiber bundles the morphism $f_{\mathfrak{M}}$ is identity, and $f_0 = (f_E, \mathrm{id})$ is called *B-morphism* [37]. Nevertheless, in case of semi-superbundles an opposite extreme situation can take place, when $f_{\mathfrak{M}}$ is a noninvertible morphism.

For each fixed $b \in \mathfrak{M}$ there exist trivializing maps $\lambda : E_b \to U \times F$ and $\lambda' : E_{f_{\mathfrak{M}}(b)} \to U' \times F'$, $f_{\mathfrak{M}}(U) \subset U'$ which lead to a map of semi-fibers h_b determined by the commutative diagram

$$\begin{array}{ccc} E_b & \xrightarrow{f_E(b)} & E'_{f_{\mathfrak{M}}(b)} \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda'} \\ U \times F & \xrightarrow{h_b} & U' \times F' \end{array} \tag{74}$$

To describe a semi-superbundle morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ locally we choose open coverings $\mathfrak{M} = \bigcup\limits_{\alpha} U_\alpha$ and $\mathfrak{M}' = \bigcup\limits_{\alpha} U'_{\alpha'}$ together with trivializations λ_α and $\lambda'_{\alpha'}$ (see (50)). Then the connection between semi-transition functions $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ (54)–(55) of two semi-superbundles $\mathcal{L}$ and $\mathcal{L}'$ can be found from the commutative diagram

$$\begin{array}{ccc} U_{\alpha\beta} \times F & \xrightarrow{\Lambda_{\alpha\beta}} & U_{\alpha\beta} \times F \\ {\scriptstyle h_\alpha}\downarrow & & \downarrow{\scriptstyle h_\beta} \\ U'_{\alpha'\beta'} \times F' & \xrightarrow{\Lambda'_{\alpha'\beta'}} & U'_{\alpha'\beta'} \times F' \end{array} \tag{75}$$

where morphisms h_α are defined by the diagram

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle \lambda_\alpha}\downarrow & & \downarrow{\scriptstyle \lambda'_{\alpha'}} \\ U_\alpha \times F & \xrightarrow{h_\alpha} & U'_{\alpha'} \times F' \end{array} \tag{76}$$

From (75) we have the relation between semi-transition functions

$$h_\alpha \circ \Lambda_{\alpha\beta} = \Lambda'_{\alpha'\beta'} \circ h_\beta \tag{77}$$

which holds valid for noninvertible h_α as well, while in the invertible case [37, 46] the equation (77) is solved with respect to $\Lambda'_{\alpha'\beta'}$, as follows $\Lambda'_{\alpha'\beta'} = h_\alpha \circ \Lambda_{\alpha\beta} \circ h_\beta^{-1}$ (it can be considered as an equivalence of cocycles). However, in general (77) is a system of superequations which should be solved by the standard [7] or extended [8] methods of superanalysis.

Let $\mathfrak{M}$ admits two trivializing coverings $\{U_\alpha, \lambda_\alpha\}$ and $\{U'_{\alpha'}, \lambda'_{\alpha'}\}$. In general they are not connected and semi-transition functions $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ are independent. However, if $\mathfrak{M}$ is the base superspace for two semi-superbundles $\mathcal{L}$ and $\mathcal{L}'$ which are connected by a B-morphism $\mathcal{L} \xrightarrow{f_0} \mathcal{L}'$, then $\Lambda_{\alpha\beta}$ and $\Lambda'_{\alpha'\beta'}$ should agree properly.

Proposition 34. *The semi-transition functions* $\Lambda_{\alpha\beta}$ *and* $\Lambda'_{\alpha'\beta'}$ *agree if there exist additional maps* $\tilde{\Lambda}_{\alpha'\beta} : U'_{\alpha'} \cap U_\beta$ *and* $\tilde{\Lambda}_{\alpha\beta'} : U_\alpha \cap U'_{\beta'}$ *connected between themselves by the relations*

$$\tilde{\Lambda}_{\alpha'\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} = \tilde{\Lambda}_{\alpha'\beta} \tag{78}$$

on $U'_{\alpha'} \cap U_\beta$ *and*

$$\tilde{\Lambda}_{\alpha\beta'} \circ \tilde{\Lambda}_{\beta'\alpha} \circ \tilde{\Lambda}_{\alpha\beta'} = \tilde{\Lambda}_{\alpha\beta'} \tag{79}$$

on $U_\alpha \cap U'_{\beta'}$ *overlaps.*

Then the agreement conditions for $\Lambda_{\alpha\beta}$ *and* $\Lambda'_{\alpha'\beta'}$ *are*

$$\tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} = \tilde{\Lambda}_{\alpha'\beta}, \tag{80}$$

$$\Lambda_{\beta\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{81}$$

$$\tilde{\Lambda}_{\gamma\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} = \tilde{\Lambda}_{\gamma\alpha'} \tag{82}$$

on triple overlaps $U'_{\alpha'} \cap U_\beta \cap U_\gamma$ *and*

$$\Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \Lambda'_{\alpha'\beta'} = \Lambda'_{\alpha'\beta'}, \tag{83}$$

$$\tilde{\Lambda}_{\beta'\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} = \tilde{\Lambda}_{\beta'\gamma}, \tag{84}$$

$$\tilde{\Lambda}_{\gamma\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \tilde{\Lambda}_{\gamma\alpha'} = \tilde{\Lambda}_{\gamma\alpha'} \tag{85}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_\gamma$ *overlaps. Then*

$$\tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} = \tilde{\Lambda}_{\alpha'\beta}, \tag{86}$$

$$\Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} = \Lambda_{\beta\gamma}, \tag{87}$$

$$\Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} = \Lambda_{\gamma\rho}, \tag{88}$$

$$\tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \Lambda_{\beta\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} = \tilde{\Lambda}_{\rho\alpha'} \tag{89}$$

on $U'_{\alpha'} \cap U_\beta \cap U_\gamma \cap U_\rho$ *and*

$$\Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \tilde{\Lambda}_{\alpha\beta'} = \Lambda'_{\alpha'\beta'}, \tag{90}$$

$$\tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} = \tilde{\Lambda}_{\beta'\gamma}, \tag{91}$$

$$\Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} = \Lambda_{\gamma\rho}, \tag{92}$$

$$\tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} \circ \Lambda_{\gamma\rho} \circ \tilde{\Lambda}_{\rho\alpha'} = \tilde{\Lambda}_{\rho\alpha'} \tag{93}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_\gamma \cap U_\rho$ *and*

$$\Lambda'_{\alpha'\beta'} \circ \Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} = \Lambda'_{\alpha'\beta'}, \tag{94}$$

$$\Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \tilde{\Lambda}_{\beta'\gamma} = \tilde{\Lambda}_{\beta'\gamma}, \tag{95}$$

$$\tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} = \tilde{\Lambda}_{\gamma'\rho}, \tag{96}$$

$$\tilde{\Lambda}_{\rho\alpha'} \circ \Lambda'_{\alpha'\beta'} \circ \Lambda'_{\beta'\gamma'} \circ \tilde{\Lambda}_{\gamma'\rho} \circ \tilde{\Lambda}_{\rho\alpha'} = \tilde{\Lambda}_{\rho\alpha'} \tag{97}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U'_{\gamma'} \cap U_{\rho}$.

Proof. Construct a sum of trivializing coverings $\{U_\alpha, \lambda_\alpha\}$ and $\{U'_{\alpha'}, \lambda'_{\alpha'}\}$ and then use (56)–(63). □

Proposition 35. *The semi-transition functions* $\Lambda_{\alpha\beta}$ *and* $\Lambda'_{\alpha'\beta'}$ *reflexively agree if there exist additional reflexive maps* $\tilde{\Lambda}_{\alpha'\beta} : U'_{\alpha'} \cap U_\beta$ *and* $\tilde{\Lambda}_{\alpha\beta'} : U_\alpha \cap U'_{\beta'}$ *connected between themselves (in addition to (78)–(79)) by the reflexive relations*

$$\tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\beta} \circ \tilde{\Lambda}_{\beta\alpha'} = \tilde{\Lambda}_{\beta\alpha'} \tag{98}$$

on $U'_{\alpha'} \cap U_\beta$ *and*

$$\tilde{\Lambda}_{\beta'\alpha} \circ \tilde{\Lambda}_{\alpha\beta'} \circ \tilde{\Lambda}_{\beta'\alpha} = \tilde{\Lambda}_{\beta'\alpha} \tag{99}$$

on $U_\alpha \cap U'_{\beta'}$ *overlaps. The reflexive semi-transition functions* $\Lambda_{\alpha\beta}$ *and* $\Lambda'_{\alpha'\beta'}$ *should satisfy (in addition to (80)–(97)) the following reflexivity agreement relations*

$$\tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} = \tilde{\Lambda}_{\alpha'\gamma}, \tag{100}$$

$$\Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda_{\gamma\beta} = \Lambda_{\gamma\beta}, \tag{101}$$

$$\tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} = \tilde{\Lambda}_{\beta\alpha'} \tag{102}$$

on $U'_{\alpha'} \cap U_\beta \cap U_\gamma$ *and*

$$\tilde{\Lambda}_{\alpha'\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} = \tilde{\Lambda}_{\alpha'\gamma}, \tag{103}$$

$$\tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} = \tilde{\Lambda}_{\gamma\beta'}, \tag{104}$$

$$\Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\gamma} \circ \Lambda'_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} = \Lambda'_{\beta'\alpha'} \tag{105}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_\gamma$ *overlaps. Then*

$$\tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} = \tilde{\Lambda}_{\alpha'\rho}, \tag{106}$$

$$\Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} = \Lambda_{\rho\gamma}, \tag{107}$$

$$\Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} = \Lambda_{\gamma\beta}, \tag{108}$$

$$\tilde{\Lambda}_{\beta\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \Lambda_{\gamma\beta} \circ \tilde{\Lambda}_{\beta\alpha'} = \tilde{\Lambda}_{\beta\alpha'} \tag{109}$$

on $U'_{\alpha'} \cap U_\beta \cap U_\gamma \cap U_\rho$ *and*

$$\tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} = \tilde{\Lambda}_{\alpha'\rho}, \tag{110}$$

$$\Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} = \Lambda_{\rho\gamma}, \tag{111}$$

$$\tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} = \tilde{\Lambda}_{\gamma\beta'}, \tag{112}$$

$$\Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \Lambda_{\rho\gamma} \circ \tilde{\Lambda}_{\gamma\beta'} \circ \Lambda'_{\beta'\alpha'} = \Lambda'_{\beta'\alpha'} \tag{113}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U_{\gamma} \cap U_{\rho}$ *and*

$$\tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} = \tilde{\Lambda}_{\alpha'\rho}, \tag{114}$$

$$\tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} = \tilde{\Lambda}_{\rho\gamma'}, \tag{115}$$

$$\Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} = \Lambda'_{\gamma'\beta'}, \tag{116}$$

$$\Lambda'_{\beta'\alpha'} \circ \tilde{\Lambda}_{\alpha'\rho} \circ \tilde{\Lambda}_{\rho\gamma'} \circ \Lambda'_{\gamma'\beta'} \circ \Lambda'_{\beta'\alpha'} = \Lambda'_{\beta'\alpha'} \tag{117}$$

on $U'_{\alpha'} \cap U'_{\beta'} \cap U'_{\gamma'} \cap U_{\rho}$.

Analogously we can define and study a principal and associated semi-superbundles with a structure semigroup.

5. Noninvertibility and semi-homotopies

Here we briefly dwell on some possibilities to extend the notion of homotopy on continuous noninvertible mappings.

A *homotopy* [29, 53] is a continuous mapping between two maps $f : X \to Y$ and $g : X \to Y$ in the space $\mathcal{C}(X, Y)$ of maps $X \to Y$ such that $\gamma_{t=0}(x) = f(x)$, $\gamma_{t=1}(x) = g(x)$. Such maps are called *homotopic*. In other words [84] a homotopy from X to Y is a continuous function $\Gamma : X \times I \to Y$ where $I = [0, 1]$ is a unit interval. For a fixed $t \in I$ one has *stages* $\gamma_t : X \to Y$ defined by $\gamma_t(x) = \Gamma(x, t)$. The relation of homotopy divides $\mathcal{C}(X, Y)$ into a set of equivalent classes $\pi(X, Y)$ called *homotopy classes* which are a set of connected components of $\mathcal{C}(X, Y)$. Therefore, $\pi(*, Y)$ ($*$ is a point) the homotopy classes correspond to connected components of Y. If $\mathcal{C}(X, Y)$ is connected, then the homotopy between $f(x)$ and $g(x)$ can be chosen as their average, i.e.

$$\gamma_t(x) = tf(x) + (1 - t)g(x). \tag{118}$$

Two maps f and g are *homotopically equivalent* if $f \circ g$ and $g \circ f$ are homotopic to the identity.

Now let X and Y are supermanifolds in some of the definitions [13, 17, 79] or semi-supermanifold in our definition given above. Then there exist a possibility to extend the notion of homotopy. The idea is in extending the definition of the parameter t. In the standard case the unit interval $I = [0, 1]$ was taken for simplicity, because any two intervals on a real axis are homeomorphic, and so they are topologically equal.

In supercase the situation is totally different. Let X and Y are defined over Λ , a commutative $\mathbb{Z}_2$-graded superalgebra admitting a decomposition into direct sum $\Lambda = \Lambda_0 \oplus \Lambda_1$ of the even Λ_0 and odd Λ_1 parts and into the direct sum $\Lambda = \mathbb{B} \oplus \mathbb{S}$ of the body $\mathbb{B}$ and soul $\mathbb{S}$ (see [74, 79] for details). The body map $\varepsilon : \Lambda \to \mathbb{B}$ can be viewed as discarding all nilpotent superalgebra generators, which gives a number part [14]. So we have three topologically disjoint cases:

1. The parameter $t \in \Lambda_0$ is even and has a body, i.e. $\varepsilon(t) \neq 0$.
2. The parameter $t \in \Lambda_0$ is even and has no body, i.e. $\varepsilon(t) = 0$.
3. The parameter $\tau \in \Lambda_1$ is odd (any odd element has no body).

The first choice can be reduced to the standard case [22, 29] by means of a corresponding homeomorphism, and such t can always be considered in the unit interval $I = [0, 1]$. However, the following two possibilities are topologically disjoint from the first one and between themselves.

Definition 36. An *even semi-homotopy* between two supermaps $f : X \to Y$ and $g : X \to Y$ is a noninvertible (in general) mapping $X \to Y$ depending on a nilpotent bodyless even parameter $t \in \Lambda_0$ and two bodyless even constants $a, b \in \Lambda_0$ such that

$$\begin{aligned} \Delta I^{ab} \gamma^{even}_{t=a} &= \Delta I^{ab} f(x), \\ \Delta I^{ab} \gamma^{even}_{t=b} &= \Delta I^{ab} g(x), \end{aligned} \tag{119}$$

where

$$\begin{gathered} \gamma^{even}_t(x) = \Gamma^{even}(x, t), \ \Gamma^{even} : X \times I^{ab} \to Y, \\ I^{ab} = [a, b], \ \Delta I^{ab} = b - a. \end{gathered} \tag{120}$$

Definition 37. An *odd semi-homotopy* between two supermaps $f : X \to Y$ and $g : X \to Y$ is a noninvertible (in general) mapping $X \to Y$ depending on a nilpotent odd parameter $\tau \in \Lambda_1$ and two odd constants $\mu, \nu \in \Lambda_1$ such that

$$\begin{aligned} \Delta \mathcal{I}^{\alpha\beta} \gamma^{odd}_{\tau=\alpha} &= \Delta \mathcal{I}^{\alpha\beta} f(x), \\ \Delta \mathcal{I}^{\alpha\beta} \gamma^{odd}_{\tau=\beta} &= \Delta \mathcal{I}^{\alpha\beta} g(x). \end{aligned} \tag{121}$$

$$\begin{gathered} \gamma^{odd}_\tau(x) = \Gamma^{odd}(x, \tau), \Gamma^{odd} : X \times \mathcal{I}^{\alpha\beta} \to Y, \\ \mathcal{I}^{\alpha\beta} = [\alpha, \beta], \ \Delta \mathcal{I}^{\alpha\beta} = \beta - \alpha. \end{gathered} \tag{122}$$

Remark. In (120) and (122) I^{ab} and $\mathcal{I}^{\alpha\beta}$ are not intervals in any sense, because among bodyless variables there is no possibility to establish an order relation [13, 14, 75], and so ΔI^{ab} and $\Delta \mathcal{I}^{\alpha\beta}$ are only notations.

Nevertheless, we can give an example of an analog of the average (118) for an odd semi-homotopy

$$(\beta-\alpha)\,\gamma_\tau^{odd}(x)=(\beta-\tau)\,f(x)+(\tau-\alpha)\,g(x) \tag{123}$$

which can satisfy some supersmooth conditions.

Remark. In (119) and (121) it is not possible to cancel the left and right hand parts by I^{ab} and $\mathcal{I}^{\alpha\beta}$ correspondingly, because the solutions for semi-homotopies γ_t^{even} and γ_τ^{odd} are viewed as equivalence relations. This is clearly seen from (123) where the division by $(\beta-\alpha)$ is impossible, nevertheless a solution for $\gamma_\tau^{odd}(x)$ can exist.

The most important property of semi-homotopies is their possible noninvertibility which follows from the nilpotency of t and τ and the definitions (119) and (121). Therefore, Y cannot be a supermanifold, it can be a semi-supermanifold only.

Conjecture 38. *It can be assumed that semi-homotopies play the same or similar role in the study of continuous properties and classification of semi-supermanifolds, as the role which play ordinary homotopies for ordinary manifolds.*

So that it is worthwhile to study their properties thoroughly and in more detail, which will be done elsewhere.

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